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Solutions of the Wave Equation

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Abstract

Waves are seen in many different applications, such as sound waves, electromagnetic waves, and ocean waves. They are typically modeled mathematically by sine and cosine functions. However, they are also modeled through partial differential equations, exhibiting aspects of position and time. This paper will be an exploration of solutions to one of these partial differential equations, called the wave equation, and will use various methods to solve boundary value problems. The Fourier method will be introduced and applied to the wave equation as well as Fourier series. These methods and applications will provide a better understanding of waves and the wave equation.

1 Waves: Overview

Waves can be defined as the result of a disturbance propagating through a medium with finite velocity. The results of any kind of measurement of a wave are called signals. Although waves can be represented as many different functions, the most common mathematical representation of waves are sine and cosine functions. These functions are infinite and periodic (repeating) and reveal the properties of the amplitude and period of the waves represented. Amplitude is the height of the wave and period is how often the wave repeats.

In Figure 1 below, the equations $y = \sin x$ and $y = \cos x$ have amplitudes $a = 1$ and periods $f = 2\pi$ for both functions. This is one of the most basic

Figure 1: $y = \sin x$ and $y = \cos x$
ways to model a wave, as it is in two dimensional space and involves only one independent variable.

Another approach for representing waves is looking at the wave signals. One dimensional waves move in the $x$ direction. The signal is going to depend on both the position $x$ and the time $t$ in which it is measured. Therefore, a wave signal is a function of two variables, which will be denoted by $y(x,t)$. To visualize the signal, select some fixed times $t_0, t_1, t_2, \ldots$, and graph on the axes of $x$ and $y(x,t_n)$. Some examples are shown in Figure 2 and Figure 3 below.

![Figure 2: A wave moving to the right with decreasing amplitude](image)

![Figure 3: A shock wave moving to the left](image)

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1Bernard Deconinck, ”Partial Differential Equations and Waves”, University of Washington, May 22, 2017, [canvas.uw.edu](http://canvas.uw.edu)
2 The Wave Equation

The wave equation is a partial differential equation that describes the properties of motion in a wave and is represented by the following equation:

\[ y_{tt}(x,t) = a^2 y_{xx}(x,t), \]  

or in other notation,

\[ \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}. \]

Here \( a^2 \) is a constant, whose value will be discussed in the derivation of the wave equation. Partial differential equations are simply equations that include several variables and derivatives of these variables with respect to one variable at a time. The partial derivatives describe the change in one variable with respect to another, while all other variables are treated as constant. In order to derive the wave equation, we will use an example of a vibrating string.

Consider a tightly stretched string, whose equilibrium position is on the \( x \) axis. The string vibrates on the \( xy \) plane. Each point \( (x,0) \) of the wave in the equilibrium position has a transverse displacement \( y = y(x,t) \) at time \( t \). Assumptions are that the displacements and slopes are small relative to the length of the string, and the movement of each point is parallel to the \( y \) axis. Assume the tension in the string is large enough to cause the string to behave as if it had perfect flexibility. Ignoring any resistance, this implies that at a point \( (x,y(x,t)) \), the part of the string to the left of the point exerts a tangential force of tension \( T \) on the part of the string to the right of the point. The magnitude of the \( x \) component of this force is denoted by \( H \), and is assumed to be constant. The \( y \) component of \( T \) is denoted by \( V(x,t) \) and \( \alpha \) denotes the angle of inclination of the string at this point.\(^2\)

\[ \begin{align*}
\text{Figure 4: Force diagram of a vibrating string} \\
\end{align*} \]

Given all the information above, visualized by Figure 4, the equations

$$\frac{-V(x, t)}{H} = \tan \alpha = y_x(x, t) \quad \text{if } V(x, t) < 0 \text{ and } y_x(x, t) > 0$$

and

$$\frac{V(x, t)}{H} = \tan (\pi - \alpha) = -\tan \alpha = -y_x(x, t) \quad \text{if } V(x, t) > 0 \text{ and } y_x(x, t) < 0$$

are obtained. Following from the above equation,

$$V(x, t) = -Hy_x(x, t), \quad \text{where } H > 0. \quad (2)$$

Ignoring all external forces, consider a segment of the string (not containing an endpoint) and let its projection onto the $x$ axis be denoted by $\Delta x$. Let the mass per unit length of the string be denoted by $\delta$. Since displacements do not have an $x$ component, that the mass of the segment is $\delta \Delta x$. Let $S$ be the tangential force exerted on the other side of the string (force that the right side exerts on the left side) and its $y$ component be $V(x + \Delta x, t)$. Let $\beta$ be the angle of inclination on the other side of the string. This implies

$$\frac{V(x + \Delta x, t)}{H} = \tan \beta$$

and therefore

$$V(x + \Delta x, t) = H y_x(x + \Delta x, t), \quad (3)$$

where $H > 0$. The acceleration of $(x, y)$ in the $y$ direction is $y_{tt}(x, t)$, or $\frac{\partial^2 y}{\partial t^2}$. Using Newton’s second law of motion, $ma = F$,

$$\delta \Delta x y_{tt}(x, t) = -Hy_x(x, t) + H y_x(x + \Delta x, t)$$

when $\Delta x$ is small. It follows that

$$y_{tt}(x, t) = \frac{H}{\delta} \lim_{\Delta x \to 0} \frac{y_x(x + \Delta x, t) - y_x(x, t)}{\Delta x} = \frac{H}{\delta} y_{xx}(x, t)$$

when the partial derivatives exist. Therefore, the function of transverse displacements in a stretched string $y(x, t)$ satisfies the wave equation:

$$y_{tt}(x, t) = a^2 y_{xx}(x, t),$$

where $a^2 = \frac{H}{\delta}$, and $a$ has the physical dimensions of velocity.

---

3 Preliminaries

3.1 Principle of Superposition

Theorem 1 If each of \( N \) functions \( u_1, u_2, ..., u_N \) satisfies a linear homogeneous differential equation \( Lu = 0 \), then every linear combination

\[
u = \sum_{n=1}^{N} c_n u_n,
\]

(4)

where the \( c \)'s are arbitrary constants, satisfies that differential equation. If each of the \( N \) functions satisfies a linear homogeneous boundary condition \( Lu = 0 \), then every linear combination also satisfies that boundary condition.

In addition, if dealing with an infinite set:

Theorem 2 Suppose that each function of an infinite set \( u_1, u_2, ... \) satisfies a linear homogeneous differential equation or boundary condition \( Lu = 0 \). Then the infinite series

\[
u = \sum_{n=1}^{\infty} c_n u_n,
\]

(5)

where the \( c_n \) are constants, also satisfies \( Lu = 0 \), provided that the series converges and is differentiable for all derivatives involved in \( L \) and provided that any required continuity condition at the boundary is satisfied by \( Lu \) when \( Lu = 0 \) is a boundary condition.

3.2 Fourier Series

Fourier series are defined as follows: Given \( f(x), c \in [-L, L] \),

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right)
\]

(6)

where

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) dx,
\]

(7)

where \( n = 0, 1, 2, ... \), and

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) dx,
\]

(8)

where \( n = 1, 2, ... \).

---

One branch of the Fourier series is called the Fourier sine series. It is obtained from the formula of the Fourier series if \( f(x) \) is odd and \( a_n = 0 \). This implies

\[
b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) dx
\]  

(9)

and

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right).
\]  

(10)

Additionally the other branch, called the Fourier cosine series can be defined as the following: If \( f(x) \) is even and \( b_n = 0 \), then

\[
a_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{n\pi x}{L} \right) dx
\]  

(11)

and

\[
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right).
\]  

(12)

### 3.3 Fourier Method

The first step of the method is called the method of separation of variables. The idea is to make the solution a combination of two functions that are not dependent on the other variable. For example, if the partial differential equation is in terms of the variables \( x \) and \( t \), we would want a solution of the form \( y = X(x)T(t) \). In this case, the \( X \) function is not dependent on \( t \), and the \( T \) function is not dependent on \( x \). The next step is to substitute this function into the partial differential equation you are trying to solve and manipulate the equation until all of the \( T \) functions are on one side, and all of the \( X \) functions are on the other side. Set each of these sides equal to their common constant, usually denoted by \(-\lambda\). Manipulate the functions of \( \lambda \) so they are set equal to 0, and plug in the boundary value conditions. Find the values of \( \lambda \) where the solution is nontrivial for both functions. The solutions for both functions can be combined, and the principle of superposition is used to find the general solution. The nonhomogeneous condition is then solved by using Fourier series.

### 4 Solutions

#### 4.1 Traveling Wave Solutions

The first to be discussed is a solution to the traveling wave. If we have a function with the form \( y(x, t) = f(x - vt) \), it represents a wave that moves at velocity \(|v|\). The direction of motion is to the right if \( v > 0 \), and to the left if \( v < 0 \). A solution of a partial differential equation where \( x \) and \( t \) are present in the form \( x - vt \) is called a traveling wave solution.
Given the wave equation \( y_{tt} = a^2 y_{xx} \) and \( a > 0 \), let \( y(x, t) = f(x - vt) \). Let \( u = x - vt \). This implies

\[
\begin{align*}
y(x, t) &= f(u) \\
y_t &= f' \\
y_{xx} &= f'' \\
y_t &= -vf' \\
y_{tt} &= (-v)^2 f'' = v^2 f''
\end{align*}
\]

Substituting into the wave equation,

\[
v^2 f'' = a^2 f''
\]

\[\Rightarrow (v^2 - a^2) f'' = 0\]

The following conclusions can be made:

\[f'' = 0 \Rightarrow f(u) = Au + B\]

\[v^2 = a^2 \Rightarrow v = \pm a\]

Therefore, our superimposed solution of the wave equation is

\[
y(x, t) = A(x - vt) + B + f_1(x - at) + f_2(x + at)\]

(13)

The superimposed solution of the wave equation is not itself a traveling wave solution, but each of the three parts of the solution are all examples of traveling wave solutions.

### 4.2 d’Alembert’s Solution

Given \( y_{tt}(x, t) = a^2 y_{xx}(x, t) \), where \( -\infty < x < \infty, t > 0 \) and \( y(x, 0) = f(x), y_t(x, 0) = 0 \), where \( -\infty < x < \infty \). Let \( u = x + at \) and \( v = x - at \). Using the chain rule of composite functions:

\[
\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial t}
\]

\[
= a \frac{\partial y}{\partial u} - a \frac{\partial y}{\partial v}
\]

Replacing \( y \) with \( \frac{\partial y}{\partial t} \).

\[
\frac{\partial^2 y}{\partial t^2} = a \frac{\partial}{\partial u} \left( \frac{\partial y}{\partial t} \right) - a \frac{\partial}{\partial v} \left( \frac{\partial y}{\partial t} \right)
\]

---


Substituting for $\frac{\partial y}{\partial t}$:

$$\frac{\partial^2 y}{\partial t^2} = a \frac{\partial}{\partial u} \left( a \frac{\partial}{\partial u} \left( \frac{\partial y}{\partial t} \right) - a \frac{\partial}{\partial v} \left( \frac{\partial y}{\partial t} \right) \right) - a \frac{\partial}{\partial v} \left( a \frac{\partial}{\partial u} \left( \frac{\partial y}{\partial t} \right) - a \frac{\partial}{\partial v} \left( \frac{\partial y}{\partial t} \right) \right)$$

$$\frac{\partial^2 y}{\partial t^2} = a^2 \left( \frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial v \partial u} + \frac{\partial^2 y}{\partial v^2} \right)$$

(15)

Using a similar method with $\frac{\partial y}{\partial x}$, it can be shown that:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial v \partial u} + \frac{\partial^2 y}{\partial v^2}$$

(16)

Plugging into the wave equation,

$$a^2 \left( \frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial v \partial u} + \frac{\partial^2 y}{\partial v^2} \right) = a^2 \frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial v \partial u} + \frac{\partial^2 y}{\partial v^2}$$

$$\Rightarrow \frac{\partial^2 y}{\partial v \partial u} = 0$$

or

$$y_{uv} = 0$$

Through integration, the general solution is

$$y = \phi(u) + \psi(v) = \phi(x + at) + \psi(x - at)$$

(17)

In order to determine $\phi$ and $\psi$, the boundary conditions are used to obtain:

$$\phi(x) + \psi(x) = f(x)$$

$$a\phi'(x) - a\psi'(x) = 0$$

By integrating the second of these equations

$$\psi(x) = \phi(x) + c$$

and plugging into the first of these equations,

$$\phi(x) = \frac{1}{2} [f(x) - c],$$

$$\psi(x) = \frac{1}{2} [f(x) + c],$$

the solution is obtained to be

$$y(x, t) = \frac{1}{2} [f(x + at) + f(x - at)]$$

(18)

This solution of the wave equation is d’Alembert’s solution.
4.3 Fourier Method Solution

Let a string have transverse displacements \( y(x, t) \) and be stretched between the points \( x = 0 \) and \( x = c \) on the \( x \) axis. With no external forces acting on it, the string has an initial displacement of \( y = f(x) \) and released at rest. The solution \( y(x, t) \) must satisfy the wave equation

\[
y_{tt}(x, t) = a^2 y_{xx}(x, t)
\]

where \( 0 < x < c \) and \( t > 0 \) and satisfy the boundary conditions \( y(0, t) = 0 \), \( y(c, t) = 0 \), \( y_t(x, 0) = 0 \), \( y(x, 0) = f(x) \), where \( f \) is continuous on the interval \( 0 \leq x \leq c \) and \( f(0) = f(c) = 0 \).

The first step of the Fourier method is the previously mentioned separation of variables method. In order to determine nontrivial solutions that satisfy the homogeneous conditions, functions of the form

\[
y = X(x)T(t)
\]

are sought. Substituting this into the wave equation,

\[
X(x)T''(t) = a^2 X''(x)T(t).
\]

When \( X(x) \) and \( T(t) \) are nonzero, the variables can be separated

\[
\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.
\]

This equality, since the left hand side does not vary with \( x \) and the right hand side does not vary with \( t \), shows that these variables must have a constant value in common, \(-\lambda\).

\[
\frac{T''(t)}{a^2 T(t)} = -\lambda, \quad \frac{X''(x)}{X(x)} = -\lambda.
\]

Substitution into our original equations and boundary conditions gives

\[
X''(x) + \lambda X(x) = 0,
\]

\[
X(0) = 0,
\]

\[
X(c) = 0,
\]

\[
T''(t) + \lambda a^2 T(t) = 0,
\]

and

\[
T'(0) = 0.
\]

Since the first equation has two boundary conditions, it might have nontrivial solutions for only some values of \( \lambda \). This is called a Sturm-Liouville problem. This problem will be tackled by looking at all possible values of \( \lambda \).

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If $\lambda = 0$, the equation becomes
\[ X''(x) = 0.\]
Its general solution is
\[ X(x) = Ax + B. \quad (20)\]
Therefore if $X(0) = 0$ and $X(c) = 0$ are to be satisfied, $A$ and $B$ must equal 0 and therefore
\[ X(x) = 0 \]
This is a trivial solution.
If $\lambda > 0$, let $\lambda = \alpha^2$ where $\alpha > 0$. The equation now becomes
\[ X''(x) + \alpha^2 X(x) = 0, \]
whose general solution is
\[ X(x) = C_1 \cos \alpha x + C_2 \sin \alpha x. \quad (21)\]
Since $X(0) = 0$, this implies that $C_1 = 0$. The condition $X(c) = 0$ implies that $C_2 \sin \alpha c = 0$, and if we want a nontrivial solution $C_2 \neq 0$. This implies that
\[ \alpha = \frac{n\pi}{c}, \]
\[ \Rightarrow X(x) = \sin \frac{n\pi x}{c}. \quad (22)\]
where $n \in \mathbb{N}$.
If $\lambda < 0$, let $\lambda = -\alpha^2$. The general solution is
\[ X(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}. \quad (23)\]
For the condition $X(0) = 0$ to be satisfied, $C_1 = -C_2$. This would imply that
\[ X(x) = C_1 (e^{\alpha x} - e^{-\alpha x}) = 2C_1 \sinh \alpha x. \]
$X(c) = 0$ requires $C_1 \sinh \alpha c = 0$, but since $\sinh \alpha c \neq 0$, $C_1 = 0$ which implies $X(x) = 0$. This is a trivial solution.
The eigenvalues are the values for which the partial differential equation has nontrivial solutions. For this example, the eigenvalues are
\[ \lambda_n = \left( \frac{n\pi}{c} \right)^2, \quad (24)\]
and their corresponding eigenfunctions are
\[ X_n(x) = \sin \frac{n\pi x}{c}. \quad (25)\]

Next, the solution of $T''(t) + \lambda a^2 T(t) = 0$ must be determined when $\lambda$ is an eigenvalue. When $\lambda = \lambda_n$,
\[ T''(t) + \left( \frac{n\pi a}{c} \right)^2 T(t) = 0, \]
\[ T'(0) = 0. \]

It follows, through a similar process, that
\[ T_n(t) = \cos \frac{n\pi at}{c}. \tag{26} \]

Therefore,
\[ y_n = X_n(x)T_n(t) = \sin \frac{n\pi x}{c} \cos \frac{n\pi at}{c}. \tag{27} \]

Using the principle of superposition, the generalized solution is
\[ y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \cos \frac{n\pi at}{c} \tag{28} \]

such that \( b_n \) is restricted so that the series is convergent and differentiable. The nonhomogeneous condition will be satisfied if
\[ f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \]

where \( 0 < x < c \). Notice that the nonhomogeneous condition must satisfy a Fourier sine series. In order to find the value of \( b_n \), the index of summation is changed to \( m \):
\[ f(x) = \sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{c}. \]

Multiply through by \( \sin \frac{n\pi x}{c} \), where \( n \) is a fixed positive integer,
\[ f(x) \sin \frac{n\pi x}{c} = \sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{c} \sin \frac{n\pi x}{c}. \]

Integrating with respect to \( x \) from 0 to \( c \),
\[ \int_0^c f(x) \sin \frac{n\pi x}{c} \, dx = \int_0^c \sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{c} \sin \frac{n\pi x}{c} \, dx \]
\[ = \sum_{m=1}^{\infty} b_m \int_0^c \sin \frac{m\pi x}{c} \sin \frac{n\pi x}{c} \, dx. \]

Using
\[ \int_0^c \sin \frac{m\pi x}{c} \sin \frac{n\pi x}{c} \, dx = \begin{cases} 0 & m \neq n \\ \frac{c}{2} & m = n \end{cases} \tag{29} \]

The equation reduces to:
\[ \int_0^c f(x) \sin \frac{n\pi x}{c} \, dx = b_n \frac{c}{2} \]
to get our solution:

\[ b_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} \, dx. \]  

(30)

Therefore, our final solution is

\[ y(x, t) = \sum_{n=1}^{\infty} 2 \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} \sin \frac{n\pi x}{c} \cos \frac{n\pi at}{c}. \]  

(31)

5 Conclusion

The methods and examples discussed in this paper are of the simplest form: linear, second order partial differential equations. Other problems involving the wave equation involve a more complicated process because they are more complex partial differential equations. For example, traveling wave solutions can be found of the sine-Gordon equation\(^{10}\), a nonlinear partial differential equation \(y_{tt} = y_{xx} - \sin y\). Additionally, traveling wave solutions can be found for the Korteweg-de Vries equation\(^{11}\), which is non linear and third-order partial differential equation \(y_t + yy_x + y_{xxx} = 0\). Some more complicated solutions that could be explored further are shock waves and rarefraction waves\(^{12}\). This introductory exploration into the wave equation and its solutions provides a stronger understanding of waves and their properties and solutions.

\(^{10}\)Bernard Deconinck, "Partial Differential Equations and Waves", University of Washington, May 22, 2017, canvas.uw.edu

\(^{11}\)Bernard Deconinck, "Partial Differential Equations and Waves", University of Washington, May 22, 2017, canvas.uw.edu